Lecture 22

Wednesday, 16 November 2022 11:33 AM

Maximal - In - Range Mechanisms

Multi - Parameter Mechanism Design :

- outcome space Ω (finite)
- bidders have private values $v_i : \Omega \to \mathbb{R}_+$
- want DSIC mechanisms that maximize SW

mechanism: given bids $b = (b_1, \dots, b_n)$,

$x(b) \in \arg\max_{\omega \in \Omega} \sum_i b_i(\omega)$, say $x(b) = \hat{\omega}$

$$p_i(b) = \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\hat{\omega})$$

(harm caused to other bidders)

this is DSIC ... problem is that if $|\Omega|$ is large, then it may be difficult to communicate bids, or compute $\max_{\omega \in \Omega} \sum_i b_i(\omega)$

Instead, as before, we can consider mechanisms that approximately maximize SW, but are simpler

Maximal - In - Range Mechanisms

Given a MPMD problem where $|\Omega|$ is large, suppose we can obtain $\hat{\Omega} \subseteq \Omega$ s.t.

(i) $|\hat{\Omega}|$ is small (or at least, easy to obtain bids)

(ii) easy to obtain $\max_{\omega \in \hat{\Omega}} \sum_i b_i(\omega)$

(iii) $\max_{\omega \in \hat{\Omega}} \sum_i b_i(\omega) \geq \alpha \max_{\omega \in \Omega} \sum_i b_i(\omega)$

Then consider the mechanism that obtains bids for outcomes in $\hat{\Omega}$, computes $x(b) \in \arg\max_{\omega \in \hat{\Omega}} \sum_i b_i(\omega)$, & payments

$$p_i(b) = \max_{\omega \in \hat{\Omega}} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\hat{\omega})$$ (where $\hat{\omega}$ is outcome chosen above)

Theorem: The MIR mechanism is DSIC, and is an α- approximation to the maximum social welfare

(proof: easy)

Example: Multi - Unit Auction

- n bidders, m identical items
- $\Omega = \{ \omega \in \mathbb{Z}_+^n : \sum_i \omega_i \leq m \}$
- for bidder i, $v_i(\omega)$ depends on # items he receives :

  $v_i(\omega) = v_i(\omega')$ if $\omega_i = \omega_i'$

  thus, bids / values $v_i : [m] \to \mathbb{R}_+$

  assume non-decreasing: $v_i(k+1) \geq v_i(k)$

  (this is called "free disposal")

Assume $n^2 | m$ ($m = kn^2$ for some $k \in \mathbb{Z}_+$)

Claim: Maximizing SW is NP- hard if m is large

(prove yourself)

To approximate SW, we'll use an MIR mechanism.

Let $b := \frac{m}{n^2}$. Consider $\hat{\Omega}$, where each bidder is assigned a multiple of b items. (called a "bundle")

Thus $\hat{\Omega} = \{ \omega \in \mathbb{Z}_+^n : \sum_i \omega_i b \leq m \}$

Claim: Given bids $\{ b_i : \hat{\Omega} \to \mathbb{R}_+ \}_{i \in N}$, the optimal allocation $x(b) \in \arg\max_{\omega \in \hat{\Omega}} \sum_i b_i(\omega)$ can be obtained in polynomial time.

(prove yourself)

Claim: Given bids $\{ b_i : \Omega \to \mathbb{R}_+ \}_{i \in N}$ that are non-decreasing, let $\omega^* \in \arg\max_{\omega \in \Omega} \sum_i b_i(\omega)$, $\hat{\omega} \in \arg\max_{\omega \in \hat{\Omega}} \sum_i b_i(\omega)$

OPT & $\widehat{\text{OPT}}$ are the respective optimal values.

Then $\widehat{\text{OPT}} \geq \frac{1}{2} \text{OPT}$.

Proof: Assume all items allocated in ω^* (ok since b_i's non-decreasing). Let $k \in N$ be the agent allocated max. items in ω^*. Then $\omega_k^* \geq \frac{m}{n}$.

Case I: $b_k(\omega_k^*) \geq \sum_{i \neq k} b_i(\omega_i^*)$

Consider the allocation $\omega' \in \hat{\Omega}$ that allocated all m items to k. Then

$$\widehat{\text{OPT}} = \sum_i b_i(\hat{\omega}) \geq \sum_i b_i(\omega') \geq b_k(\omega^*) \geq \frac{1}{2} \sum_i b_i(\omega^*) = \frac{\text{OPT}}{2}$$

Case II: $b_k(\omega_k^*) < \sum_{i \neq k} b_i(\omega_i^*)$

For all $i \neq k$, let $\lambda_i := \left\lceil \frac{\omega_i^*}{m/n^2} \right\rceil$

Consider $\omega' \in \hat{\Omega}$: $\omega_k' = 0$

$\omega_i' = \lambda_i \cdot \frac{m}{n^2}$

(if there are items remaining, # remaining items must be a multiple of m/n^2. Assign these to agent 1).

Note that this is a valid assignment :

- at least m/n items taken from k,
- $(n-1)$ agents get at most m/n^2 items.

Finally,

$$\widehat{\text{OPT}} = \sum_i b_i(\hat{\omega}) \geq \sum_i b_i(\omega')$$

$$= \sum_{i \neq k} b_i(\omega') \geq \sum_{i \neq k} b_i(\omega^*) \geq \frac{1}{2} \text{OPT} \quad \blacksquare$$